COUNTING PRINCIPLES, PERMUTATION AND COMBINATION

HEMANGA DUTTA

M.Sc IN MATHEMATICS

ROLL NO. : MSM14039

TEZPUR UNIVERSITY

OUTLINE

In this presentation, we are going to discuss about,

1. **Four Basic Counting Principles**
   1. Addition
   2. Multiplication
   3. Subtraction
   4. Division
2. **Permutation**
   1. Permutation of Multiset
3. **Combination**
   1. Combination of multiset

Four Basic Counting Principles

There are **FOUR** basics counting principle .they are,

1. **Addition**
2. **Multiplication**
3. **Subtraction**
4. **Division**

ADDITION PRINCIPLE : *Suppose that a set S is partitioned into parts $S_1, S_2, \ldots, S_m$. The number of of objects in S can be determined by finding the number of objects in each parts, and adding the numbers so obtaine:*

$$|S| = |S_1| + |S_2| + \cdots + |S_m|$$

ADDITION PRINCIPLE : *Suppose that a set S is partitioned into parts S_1,S_2,...,S_m. The number of of objects in S can be determined by finding the number of objects in each parts, and adding the numbers so obtaine:*

$$|S| = |S_1| + |S_2| + \cdots + |S_m|$$

Example : A student wishes to take either a mathematics course or a biology course, but not both. If there are 4 mathematics courses and 3 biology courses for which the student has the necessary prerequisites, then the student can choose a course to take in 4+3=7 ways.

MULTIPLICATION PRINCIPLE : *Let S be a set of ordered pair (a, b) of objects, where the first object a comes from a set of size p, and for each choice of a there are q choices for object b. Then the size of S is $p \times q$:*

$$|S| = p \times q.$$

MULTIPLICATION PRINCIPLE : *Let S be a set of ordered pair (a, b) of objects, where the first object a comes from a set of size p, and for each choice of a there are q choices for object b. Then the size of S is $p \times q$:*

$$|S| = p \times q.$$

The multiplication principle is actually a consequence of the addition principle. Let $a_1, a_2, \ldots, a_p$ be the p different choices for the object a. We partition S into parts $S_1, S_2, \ldots, S_p$ where S_i is the set of ordered pairs in S with first object a_i, (i=1,2,...,p). The size of each $|S_i|$ is q; hence, by the addition principle,

$$\begin{aligned} |S| &= |S_1| + |S_2| + \cdots + |S_p| \\ &= q + q + \cdots + q \\ &= p \times q \end{aligned}$$

Example 1: A student is to take two courses. The first meets at any one of 3 hours in the morning, and the second at any one of 4 hours in the afternoon. The number of schedules that are possible for the student is $3 \times 4 = 12$.

Example 1: A student is to take two courses. The first meets at any one of 3 hours in the morning, and the second at any one of 4 hours in the afternoon. The number of schedules that are possible for the student is $3 \times 4 = 12$.

Example 2: Chalk comes in three different length, 8 different colors, and 4 different diameters. How many different kind of chalk are there ?

Ans. To determine a piece of chalk we carry out 3 different task : choose a length, choose a color, choose a diameter. By the multiplication principle, there are $3 \times 8 \times 4 = 96$ different kinds of chalk.

SUBTRACTION PRINCIPLE : *Let A be a set and let U be a larger set containing A. Let*

$$\overline{A} = \{x \in U : x \notin A\}$$

be the complement of A in U. Then the number $|A|$ of objects in A is given by the rule

$$|A| = |U| - |\overline{A}|.$$

SUBTRACTION PRINCIPLE : *Let A be a set and let U be a larger set containing A. Let*

$$\overline{A} = \{x \in U : x \notin A\}$$

be the complement of A in U. Then the number $|A|$ of objects in A is given by the rule

$$|A| = |U| - |\overline{A}|.$$

In applying the subtraction principle, the set U is usualy some natural set consisting of all the objects under discussion (the so-called universal set). Using the subtraction principle makes sense only if it is esier to count the number of objects in U and $|\overline{A}|$ then to count the number of objects in A.

Four Basic Counting Principles

Example 1: Count the number of integers between 1 and 600, which are not divisible by 6.

Ans . Here U = the whole set = 600

$A = \{x : x$ is between 1 and 600 and not divisible by 6 $\}$

$$\begin{aligned} \overline{A} &= \{x : x \notin A\} \\ &= \{x : x/6\} \\ &= 600/6 \\ &= 100 \\ therefore, & \\ |A| &= |U| - \overline{|A|} \\ &= 600 - 100 \\ &= 500 \end{aligned}$$

DIVISON PRINCIPLE : *Let S be a finite set that is partitioned into k parts in such a way that each part contains the same number of objects. Then the number of parts in the partition is given by the rule*

$$k = \frac{|S|}{number \;\; of \;\; objects \;\; in \;\; a \;\; part}$$

DIVISON PRINCIPLE : *Let S be a finite set that is partitioned into k parts in such a way that each part contains the same number of objects. Then the number of parts in the partition is given by the rule*

$$k = \frac{|S|}{number \;\; of \;\; objects \;\; in \;\; a \;\; part}$$

Thus, we can determine the number of parts if we know the number of objects in S and the common value of the number of objects in the parts.

Example : There are 740 pigeons in a collection of pigeonholes. If each pigeonhole contains 5 pigeons, the number of pigeonholes equals

$$740/5 = 148$$

Permutations

Let r be a positive integer. By an *r-permutation* of a set S of n elements, we understand an ordered arrangement of r of the n elements.

Let r be a positive integer. By an *r-permutation* of a set S of n elements, we understand an ordered arrangement of r of the n elements. If

$$S = \{a, b, c\}$$

then the 1-*permutation* of S are

$$a \qquad\qquad b \qquad\qquad c$$

Permutations

Let r be a positive integer. By an *r-permutation* of a set S of n elements, we understand an ordered arrangement of r of the n elements. If

$$S = \{a, b, c\}$$

then the 1-*permutation* of S are

$$a \qquad b \qquad c$$

the six 2-*permutations* of S are

$$ab \qquad ac \qquad ba \qquad bc \qquad ca \qquad cb$$

Permutations

Let r be a positive integer. By an *r-permutation* of a set S of n elements, we understand an ordered arrangement of r of the n elements. If

$$S = \{a, b, c\}$$

then the 1-*permutation* of S are

$$a \qquad b \qquad c$$

the six 2-*permutations* of S are

$$ab \qquad ac \qquad ba \qquad bc \qquad ca \qquad cb$$

the six 3-*permutations* of S are

$$abc \qquad acb \qquad bac \qquad bca \qquad cab \qquad cba$$

Theorem 1 : For n and r positive integers with $r \leq n$.

$$P(n, r) = n \times (n-1) \times \ldots\ldots \times (n-r+1)$$

Theorem 1 : For n and r positive integers with $r \leq n$.

$$P(n, r) = n \times (n-1) \times ...... \times (n-r+1)$$

Results : For a non-negative integer n, we define $n!$ by

$$n! = n \times (n-1) \times \cdots \times 2 \times 1$$

with the convention that 0!=1. We may then write

$$P(n, r) = \frac{n!}{(n-r)!}$$

Theorem 1 : For n and r positive integers with $r \leq n$.

$$P(n,r) = n \times (n-1) \times ...... \times (n-r+1)$$

Results : For a non-negative integer n, we define $n!$ by

$$n! = n \times (n-1) \times \cdots \times 2 \times 1$$

with the convention that 0!=1. We may then write

$$P(n,r) = \frac{n!}{(n-r)!}$$

Example 1 : The number of 4-letter "words" that can be formed by using each of letters a, b, c, d, e at most once is $P(5,4)$, and this equals 5!/ (5-4)! $=$ 120. The number of 5-letter words equals $P(5,5)$, which is also 120 .

Theorem 2: The number of circular r-permutations of a set of n elements is given by

$$\frac{P(n,r)}{r} = \frac{n!}{r \times (n-r)!}$$

In particular, the number of circular permutations of n elements is $(n-1)!$

Theorem 2: The number of circular r-permutations of a set of n elements is given by

$$\frac{P(n,r)}{r} = \frac{n!}{r \times (n-r)!}$$

In particular, the number of circular permutations of n elements is $(n-1)!$

Example :

Permutations

There are some results in permutation. they are

1. $P(n, n) = n!$
2. $P(n, 0) = \frac{n!}{(n-0)!} = 1$
3. $P(n, 1) = 1$
4. $P(n, n-1) = n!$

If S is a multiset, an *r-permutation* of S is an ordered arrangement of r of the objects of S. If the total number of objects of S is n (counting repetitions), then an n-permutation of S will also be called a *permutation* of S.

Permutation of Multiset

If S is a multiset, an *r-permutation* of S is an ordered arrangement of r of the objects of S. If the total number of objects of S is n (counting repetitions), then an n-permutation of S will also be called a *permutation* of S. For example, if $S = \{2.a, 1.b, 3.c\}$, then

$$acbc \qquad cbcc$$

are 4-permutations of S, while

$$abccca$$

is a permutation of S.

If S is a multiset, an *r-permutation* of S is an ordered arrangement of r of the objects of S. If the total number of objects of S is n (counting repetitions), then an n-permutation of S will also be called a *permutation* of S. For example, if $S = \{2.a, 1.b, 3.c\}$, then

$$acbc \qquad cbcc$$

are 4-permutations of S, while

$$abccca$$

is a permutation of S.

Remark : The multiset S has no 7-permutations since 7 is greater then $2+1+3=6$, the number of objects of S.

Permutation of Multiset

We count the number of r-permutations of a multiset S

1. each of whose repetition number is infinite.

Permutation Of Multiset

We count the number of r-permutations of a multiset S

1. each of whose repetition number is infinite.
2. each of whose repetition number is finite.

Permutation of Multiset

Theorem 1: Let S be a multiset with objects of k different types, where each has an infinite repetition number. Then the number of r-permutations of S is k^r.

Permutation of Multiset

Theorem 1: Let S be a multiset with objects of k different types, where each has an infinite repetition number. Then the number of r-permutations of S is k^r.

Example 1: What is the number of ternary numerals with at most 4 digits.
Ans. The answer to this question is the number of 4-permutations of the multiset $\{\infty.0, \infty.1, \infty.2\}$ or of the multiset $\{4.0, 4.1, 4.2\}$. By previous Theorem, this number equals $3^4 = 81$.

Permutation of Multiset

Theorem 2: Let S be a multiset with objects of k different types with finite repetition numbers $n_1, n_2, \ldots, n_k$, respectively. Let the size of S be $n = n_1 + n_2 + \cdots + n_k$. Then the number of permutations of S equals

$$\frac{n!}{n_1!n_2!\ldots n_k!}.$$

Theorem 2: Let S be a multiset with objects of k different types with finite repetition numbers $n_1, n_2, \ldots, n_k$, respectively. Let the size of S be $n = n_1 + n_2 + \cdots + n_k$. Then the number of permutations of S equals

$$\frac{n!}{n_1! n_2! \ldots n_k!}.$$

Example 2: The number of permutation of the letters in the word MISSISSIPPI is

$$\frac{11!}{1!4!4!2!},$$

since this number equals the number of permutations of the multiset $\{1.M, 4.I, 4.S, 2.P\}$

Combinations

Let r be a non negative integer. By an *r-combination* of a set S of n elements, we understand an unordered selection of r of the n objects of S.

Combinations

Let r be a non negative integer. By an *r-combination* of a set S of n elements, we understand an unordered selection of r of the n objects of S. If $S = \{a, b, c, d\}$, then

$$\{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}$$

are the four 3-*cobination* of S. We denote by $\binom{n}{r}$ the number of r-combinations of an n-element set.

Let r be a non negative integer. By an *r-combination* of a set S of n elements, we understand an unorderd selection of r of the n objects of S. If $S = \{a, b, c, d\}$, then

$$\{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}$$

are the four 3-*cobination* of S. We denote by $\binom{n}{r}$ the number of r-combinations of an n-element set. Obviously,

$$\binom{n}{r} = 0 \qquad \textit{if} \quad r > n.$$

also,

$$\binom{0}{r} = 0 \qquad \textit{if} \quad r > 0.$$

Combinations

the following additional facts are readily seen to be true for each non-negative integer n:

1. $\binom{n}{0} = 1$
2. $\binom{n}{1} = n$
3. $\binom{n}{n} = 1$
4. $\binom{0}{0} = 1$

Combinations

Theorem : For $0 \le r \le n$,

$$P(n, r) = r! \binom{n}{r}.$$

Hence,

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

Combinations

Theorem : For $0 \leq r \leq n$,

$$P(n,r) = r!\binom{n}{r}.$$

Hence,

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

Example : Twenty-five points are chosen in the plane so that no three of them are collinear. How many straight lines do they determine?

Ans. Since no three of the points lie on a line, every pair of points determines a unique straight line. Thus, the number of straight lines determined equals the number of 2-combinations of a 25-element set, and this is given by

$$\binom{25}{2} = \frac{25!}{2!23!} = 300.$$

Combinations of Multiset

If S is a multiset, then an *r-combination* of S is an unordered selection of r of the objects of S. Thus, an r-combination of S is itself a multiset, a submultiset of S.

1. If S has n objects, then there is is only one n-combination of S, namely, S itself.
2. If S contains objects of k different types, then there are k 1-combinations of S.

If S is a multiset, then an *r-combination* of S is an unordered selection of r of the objects of S. Thus, an r-combination of S is itself a multiset, a submultiset of S.

1. If S has n objects, then there is is only one n-combination of S, namely, S itself.
2. If S contains objects of k different types, then there are k 1-combinations of S.

Example : If $S=\{2.a, 1.b, 3.c\}$, then the 3-combinations of S are

$$\{2.a, 1.b\}, \qquad \{2.a, 1.c\}, \qquad \{1.a, 1.b, 1.c\},$$

$$\{1.a, 2.c\}, \qquad \{1.b, 2.c\}, \qquad \{3.c\}.$$

Combinations of Multiset

Theorem : Let S be a multiset with objects of k types, each with an infinite repetition number. Then the number of r-combinations of S equals

$$\binom{r+k-1}{r} = \binom{r+k-1}{k-1}$$

Theorem : Let S be a multiset with objects of k types, each with an infinite repetition number. Then the number of r-combinations of S equals

$$\binom{r+k-1}{r} = \binom{r+k-1}{k-1}$$

Example : A bakery boasts 8 varieties of doughnuts. If a box of doughnuts contain 1 dozen, how many different options are there for a box of doughnuts?

Ans. It is assumed that the bakery has on hand a large number (at least 12) of each variety. This is a combination problem, since we assume the of order of the doughnuts in a box is a box is irrelevant for the purchaser's purpose. The number of different options for boxes equals the number of 12-combinations of a multiset with objects of 8 types, each having an infinite repetition number. This number equals

$$\binom{12+8-1}{12} = \binom{19}{12}$$

Conclusion

We have discuss these following topics :

1. Using the Counting Principles
2. Finding the simple way of permutation
3. Solving the Problem of Permutation
4. Finding the simple way of permutation of Multiset
5. Finding the simple way of Combinations
6. Solving the Problem of Combinations

References

Richard A. Brualdi: Introductory Combinatorics, Fourth edition. Madison: University of Wisconsin, 2008.

C.L.Liu: Elements of Discrete Mathemathics, Second Edition. Urbana-Champaign: University of Illinois, 2000

Ralph P.Grimaldi: Discrete and Combinatorial Mathematics, Fifth edition.USA: Rose–Hulman Institute of Technology,2003


Thank you!